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THE CONTACT PROBLEM OF THE DISCRETE FITTING OF AN INHOMOGENEOUS VISCOELASTIC AGEING CYLINDER WITH A SYSTEM OF RIGID COLLARS†

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The axially symmetric contact problem of the interaction of an inhomogeneous ageing viscoelastic cylindrical body with an arbitrary finite system of fitted rigid elements is considered. Account is taken of the fact that the collars are not fitted or removed at the same time, which is dictated, for example, by the particular features of the installation of engineering structures, as well as the properties of the age and structural inhomogeneities of the deforming body itself due to manufacturing processes or the erection of real objects. A formulation of the problem and its system of resolvent bidimensional integral equations are given. A solution of the system is constructed. A numerical analysis of a number of actual processes is carried out and the mechanisms of both the individual as well as the combined effect of the main factors on the characteristics of the contact interaction are investigated.

1. FORMULATION AND RESOLVENT EQUATIONS OF THE CONTACT PROBLEM

LET US investigate the process of the sequential fitting of rigid collars to a bilayer hollow cylinder, the layers of which are made out of different viscoelastic ageing materials at different instants of

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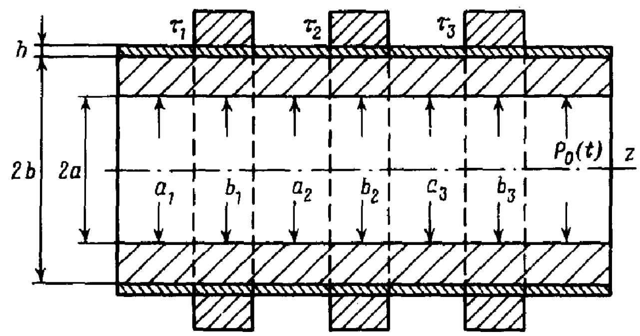

FIG. 1.

time [1, 2]. We shall also take account of the possibility that some of the collars are removed. Let each ith collar be slipped onto a segment of the cylinder $a_i \leqslant x \leqslant b_i$ ($i = 1, 2, \ldots, n$) without friction. The tightness of the ith collar is δ_i^0, the profile of its internal surface is $g_i[z-(a_i+b_i)/2]$, the instant when it is fitted is τ_i and the instant when it is removed is τ_i^0. The external layer of the cylinder which is in immediate contact with the collars is manufactured at an instant of time τ_1^* and has a thickness h. The internal layer is made at an instant of time τ_2^* and its geometrical dimensions are characterized by the values of the radii a and b ($a<b$). It is assumed that $b_i - a \gg h$, $b \gg h$ and, furthermore, that the compliance of the elements of the external, relatively thin layer and the elements of the internal layer of arbitrary thickness is of the same order or that the external layer is more compliant [3, 4]. There is smooth contact between the layers. The internal surface of the bilayer cylinder is subjected to the action of a uniform pressure $P_0(t)$ [5] which is applied at an instant of time $\tau_0 \geqslant \max[\tau_1^*, \tau_2^*]$ (Fig. 1), or a rigid insert, which creates conditions for a smooth and ideal contact to be set up in the cylinder from the very beginning. It is assumed that the distance between the collars and the ends of the cylinder is quite large and that the ends themselves are covered by rigid caps which eliminate their axial motion.

Let us now consider the first version of the formulation of the problem when, during the fitting of the cylinder with rigid collars, a certain pressure $P_0(t)$, which varies with time, is applied to it. On the basis of [5], we obtain a system of two-dimensional integral equations which describes the successive installation (removal) of the rigid elements in the following form (also, see [2]):

$$
\begin{gathered}
(1-\nu_1^2)\, h\, (\mathbf{I}-\mathbf{L}_1(\tau_i, t))\, q_i(z,t)/E_1^0(t) + \\
+\frac{2(1-\nu_2^2)}{\pi}(\mathbf{I}-\mathbf{L}_2(\tau_1,t)) \sum_{j=1}^{n} \int_{a_j}^{b_j} \frac{q_j(\xi,t)}{E_2^0(t)}\, k^\circ(z,\xi)\, d\xi = \\
= \delta_i^\circ - g_i[z-(a_i+b_i)/2] + H(t-\tau_0)(\mathbf{I}-\mathbf{L}_1(\tau_0,t)) \times \\
\times (\mathbf{I}+\mathbf{N}_3(\tau_0,t))\, \theta_1^i(t)\, (\mathbf{I}-\mathbf{L}_2(\tau_0,t))\, P_0(t)/E_2^0(t) \qquad (1.1) \\
(\theta_1^i(t) = \theta_1(t),\ a_i \leqslant z \leqslant b_i,\ i = 1, \ldots, n) \\
\tau_i^0 < t < \tau_i:\ q_i(z,t) = 0,\ \delta_i^0 = 0,\ \theta_1^i(t) = 0,\ g_i[z-(a_i+b_i)/2] = 0 \\
E_k^0(t) = E_k(t-\tau_k^*),\quad C_k^0(t,\tau) = C_k(t-\tau_k^*,\ \tau-\tau_k^*), \\
K_0^{(k)}(t,\tau) = K^{(k)}(t-\tau_k^*,\ \tau-\tau_k^*) \\
\mathbf{L}_k(s,t)\,\varphi(t) = \int_s^t \varphi(\tau)\, K_0^{(k)}(t,\tau)\, d\tau, \quad \mathbf{N}_3(\tau_0,t)\,\varphi(t) = \int_{\tau_0}^t \varphi(\tau)\, R_3(t,\tau)\, d\tau
\end{gathered}
$$

$$
\begin{gathered}
K^{(k)}(t, \tau) = E_k(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{E_k(\tau)} + C_k(t, \tau)\right] \\
\theta_1(t) = d_1\theta(t), \;\; \theta(t) = [d_2E_1^0(t)/E_2^0(t) - d_3]^{-1} \\
d_1 = 2(1-\nu_1)(h+b)^{-1}(1+\nu_2)[(1-2\nu_2)b - a^2b^2(b^2-a^2)^{-1}d_4] \\
d_2 = (1+\nu_1)^{-1}[(b+h)^{-2} - b^{-2}](1+\nu_2)a^2b^2(b^2-a^2)^{-1}d_4 \\
d_3 = (1-2\nu_1)(b+h)^{-2}b + b^{-1}, \;\; d_4 = (1-2\nu_2)ba^{-2} + b^{-1} \\
K_3(t, \tau) = [d_2K_0^{(2)}(t, \tau)E_1^0(\tau)/E_2^0(\tau) - d_3K_0^{(1)}(t, \tau)]\theta(t)
\end{gathered}
$$

$$
k^\circ(z, \xi) = \int_0^\infty \frac{L[(b-a)\alpha]}{\alpha} \cos\alpha(z-\xi)\, d\alpha \tag{1.2}
$$

$$
\begin{gathered}
L[(b-a)\alpha] = \alpha[\alpha^2aA_{01}^2 - a^0A_{11}^2 - a^{-1}]S_1^{-1} \\
S_1 = a^0b^{-1} + b^0a^{-1} + a^0b^0A_{11}^2 - b^0\alpha^2aA_{01}^2 + \\
+ \alpha^4abA_{00}^2 - a^0b\alpha^2A_{10}^2 \\
A_{ij} = I_i(\alpha a)K_j(\alpha b) - (-1)^{i+j}I_j(\alpha b)K_i(\alpha a)
\end{gathered} \tag{1.3}
$$

where $q_i(z, t)$ are the contact stresses under the ith collar and $K^{(k)}(t, \tau)$, $C_k(t, \tau)$, $E_k(t)$ and ν_k are the creep kernels, the measures of the creep, the instantaneous elastic moduli and the constant Poisson's ratios, respectively, of the materials of the outer ($k = 1$) and inner ($k = 2$) layers, $R_3(t, \tau)$ is the resolvent of the kernel $K_3(t, \tau)$, $H(t)$ is the Heaviside function and $I_m(\alpha)$, $K_m(\alpha)$ ($m = 0, 1$) are Bessel functions of imaginary argument. The presence of an internal pressure in the cylinder is taken into account by the last term on the right-hand side of the system of equations (1.1).

The systems of integral equations of a further two versions of the formulation of the problem (with a rigid insert) can be obtained by putting $P_0(t) = 0$ in (1.1) and taking the corresponding expressions for the kernel of the contact problem $k(z, \xi)$. The general form of formula (1.2) is preserved here but relationship (1.3) changes. So, in the case of a rigid insert† under conditions of smooth contact

$$
L[(b-a)\alpha] = \alpha A_{11}^2S_2^{-1}, \;\; S_2 = \alpha^2bA_{10}^2 - b^\circ A_{11}^2 - b^{-1}
$$

and, under conditions of coupling

$$
\begin{gathered}
L[(b-a)\alpha] = [a^{-1} + 4(1-\nu_2)\alpha A_{01}A_{11} - a\alpha^2(A_{01}^2 - A_{11}^2)]S_3^{-1} \\
S_3 = 8(1-\nu_2^2)\alpha(A_{00}A_{11} - A_{01}A_{10}) + ab^\circ\alpha(A_{01}^2 - A_{11}^2) + \\
+ ab\alpha^3(A_{10}^2 - A_{00}^2) - 4(1-\nu_2)(b\alpha^2A_{00}A_{10} + b^\circ A_{01}A_{11}) + \\
+ 2(1-\nu_2)(1-2\nu_2)\alpha^{-1}a^{-1}b^{-1} - (ab^{-1} + ba^{-1})\alpha
\end{gathered}
$$

We note that the properties of the kernels of the contact problems under consideration and of the kernels of planar contact problems [3, 6] are similar. In particular, $L(\alpha)\alpha^{-1} > 0$ ($|\alpha| < \infty$). With respect to the creep kernels $K^{(k)}(t, \tau)$, it is assumed that they are continuous or weakly singular.

†V. A. Chernysh, The action of normal and tangential loads on single- and multi-layer hollow circular cylinders. Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1988, 40 pp.; deposited in the All-Union Institute for Scientific and Technical Information (VINITI), No. 6784-B88, 31.08.88.

2. SOLUTION OF THE PRINCIPAL SYSTEM OF EQUATIONS

Let us make a change of variables in the system of equations (1.1) in accordance with the formulas

$$
\begin{gathered}
z^* = (2z - a_i - b_i)/(b_i - a_i) \quad (a_i \leqslant z \leqslant b_i) \\
\xi^* = (2\xi - a_i - b_i)/(b_i - a_i) \quad (a_i \leqslant \xi \leqslant b_i) \\
t^* = t\tau_1^{-1}, \ \tau^* = \tau\tau_1^{-1}, \ \tau_i^* = \tau_i\tau_1^{-1}, \ [\tau_i^\circ]^* = \tau_i^\circ\tau_1^{-1} \\
[\tau_k^*]^* = \tau_k^*\tau_1^{-1} \ (k = 1, 2), \ \tau_0^* = \tau_0\tau_1^{-1}, \ u = (b - a)\alpha \\
\frac{2(b-a)}{b_1 - a_1} = \lambda, \quad \frac{a_j + b_j}{b_1 - a_1} = \eta_j, \quad \frac{b_j - a_j}{b_1 - a_1} = \zeta_j \\
k^{ij}(z^*, \xi^*) = \frac{\sqrt{\zeta_i\zeta_j}}{\pi} k\left(\frac{\zeta_i z^* + \eta_i - \zeta_j\xi^* - \eta_j}{\lambda}\right) = \frac{\sqrt{\zeta_i\zeta_j}}{\pi} k\left(\frac{z-\xi}{b-a}\right) \\
k\left(\frac{z-\xi}{b-a}\right) = k^\circ(z, \xi), \quad k(s) = \int_0^\infty \frac{L(u)}{u}\cos su\, du, \mathbf{I}^* = \mathbf{I} \\
q^i(z^*, t^*) = 2q_i(z, t)(1 - \nu_2^2)\sqrt{\zeta_i}/E_2^\circ(t), \ g^i(z^*) = 2g_i[z - \\
- (a_i + b_i)/2]\sqrt{\zeta_i}(b_1 - a_1)^{-1} \\
P_0^i(t^*) = 2P_0(t)(1 - \nu_2^2)\sqrt{\zeta_i}/E_2^\circ(t), \quad \theta_0^i(t^*) = \theta_1^i(t)(1 - \\
- \nu_2^2)^{-1}(b_1 - a_1)^{-1} \\
c(t^*) = (1 - \nu_1^2)E_2^\circ(t)h[(1 - \nu_2^2)E_1^\circ(b_1 - a_1)]^{-1}, \ R_3^*(t^*, \tau^*) = \\
= R_3(t, \tau)\tau_1 \\
K_0(t^*, \tau^*) = K_0^{(1)}(t, \tau)\tau_1, \ K_1(t^*, \tau^*) = [E_1^\circ(t)/E_1^\circ(\tau)] \times \\
\times K_0^{(1)}(t, \tau)\tau_1[E_2^\circ(\tau)/E_2^\circ(t)] \\
K_2(t^*, \tau^*) = K_0^{(2)}(t, \tau)\tau_1, \ \delta^i = 2\delta_i^\circ\sqrt{\zeta_i}(b_1 - a_1)^{-1} \\
\mathbf{L}_m^*(s, t)f(t) = \int_s^t f(\tau)K_m(t, \tau)\,d\tau \quad (m = 0, 1, 2) \quad \mathbf{N}^*(s, t)f(t) = \\
= \int_s^t f(\tau)R_3^*(t, \tau)\,d\tau \\
f^i(z^*, t^*) = \delta^i - g^i(z^*) + H(t^* - \tau_0^*)(\mathbf{I}^* - \mathbf{L}_0^*(\tau_0^*, t^*)) \times \\
\times (\mathbf{I}^* + \mathbf{N}^*(\tau_0^*, t^*))\theta_0^i(t^*)(\mathbf{I}^* - \mathbf{L}_2^*(\tau_0^*, t^*))P_0^i(t^*), \ |z^*| \leqslant 1, \\
|\xi^*| \leqslant 1
\end{gathered} \tag{2.1}
$$

and, on omitting the asterisks in the notation for all of the quantities apart from the operators, we shall have the following system of equations:

$$
c(t)(\mathbf{I}^* - \mathbf{L}_1^*(\tau_1, t))q^i(z, t) + (\mathbf{I}^* - \mathbf{L}_2^*(1, t))\sum_{j=1}^n \mathbf{A}_{ij}^* q^j(z, t) = f^i(z, t) \tag{2.2}
$$

$$
\mathbf{A}_{ij}^* v(z) = \int_{-1}^1 k^{ij}(z, \xi)v(\xi)\,d\xi
$$

$$
\tau_i^0 < t < \tau_i: \ q^i(z, t) = 0, \ f^i(z, t) = 0
$$

Relationships (2.2) specify the stepwise process for solving contact problems in the case of cylindrical bodies and, at each step, can be reduced to a single operator equation in a functional vector space (summation is carried out over repeated superscripts and they run over all integral values from 1 to n) [2, 7]:

$$
\begin{gathered}
c(t)(\mathbf{I}^* - \mathbf{L}_1^*)\,\mathbf{q}(z,t) + (\mathbf{I}^* - \mathbf{L}_2^*)\,\mathbf{A}^*\mathbf{q}(z,t) = \boldsymbol{\varkappa}(z,t) \\
(|z| \leqslant 1,\ t \in [\tau_r,\ \tau_{r+1}]) \\
\mathbf{L}_k^* = \mathbf{L}_k^*(\tau_r,\ t)\ (k = 1,\ 2),\ \mathbf{q}(z,\ t) = q^i(z,\ t)\,i^i \\
\boldsymbol{\varkappa}(z,t) = \varkappa^i(z,t)\,\mathbf{i}^i,\quad \mathbf{A}^*\mathbf{a}(z) = (\mathbf{K}(z,\xi),\mathbf{a}(\xi)) = \\
= \int_{-1}^{1} \mathbf{K}(z,\ \xi)\cdot\mathbf{a}(\xi)\,d\xi = \int_{-1}^{1} k^{ij}(z,\xi)\,a^j(\xi)\,i^i d\xi \\
\mathbf{K}(z,\xi) = k^{ij}(z,\xi)\,\mathbf{i}^i\mathbf{i}^j,\quad \mathbf{K}(z,\xi) = \mathbf{K}^T(\xi,z),\quad \mathbf{a}(\xi) = a^i(\xi)\,\mathbf{i}^i
\end{gathered}
\tag{2.3}
$$

where $\mathbf{q}(z,t)$ is a vector function of the contact pressures which is continuous with respect to t with values from $L_2([-1,1],V)$ $((L_2[-1,1],V)$ is a Hilbert space of vector functions, the components of which have integrable squares in the interval $[-1,1]$, $\mathbf{K}(z,\xi)\in \mathbf{L}_2([-1,1],V)$ where $\mathbf{L}_2([-1,1],V)$ is a Hilbert space of tensor functions of two variables, the components of which are integrable together with their squares in the square $\{|z|\leqslant 1,\ |\xi|\leqslant 0\})$, $\mathbf{i}^k$ is an orthonormalized algebraic vector basis of an n-dimensional Euclidean space V and $\boldsymbol{\varkappa}(z,t)$ is a vector function which is continuous with respect to t and takes account of the tension of the collars, profile of their internal surface, the effect of the pressure which is imposed and, also, the distortion of the cylinder due to the creep of the material (the structure $\boldsymbol{\varkappa}(z,t)$ will be considered in greater detail below using an actual example).

It may be asserted on the basis of [2] that the operator $\mathbf{A}^*$ is completely continuous and self-adjoint from $L_2([-1,1],V)$ into $L_2([-1,1],V)$. The positive definiteness of $\mathbf{A}^*$ can be established on the basis of the definition of the positive definiteness of an operator taking account of the fact that $L(u)u^{-1}>0$ $(|u|<\infty)$ and the fact that a function with a Fourier transform which is equal to zero is equal to zero almost everywhere [8].

In constructing the solution of Eq (2.3), we shall make use of the following expansions:

$$
\mathbf{q}(z,t) = \sum_{i=0}^{\infty} \omega_i(t)\,\boldsymbol{\varphi}_i(z),\quad \boldsymbol{\varkappa}(z,t) = \sum_{i=0}^{\infty} \varkappa_i(t)\,\boldsymbol{\varphi}_i(z)
\tag{2.4}
$$

where $\varphi_i(z)$ are orthonormalized characteristic vector functions of the operator $\mathbf{A}^*$ which correspond to its characteristic numbers α_i, that is,

$$
\mathbf{A}^*\boldsymbol{\varphi}_i(z) = \alpha_i^\circ\boldsymbol{\varphi}_i(z)\ (t = 0,\ 1,\ \ldots)
\tag{2.5}
$$

On substituting series (2.4) into (2.3) and taking account of the spectral relationship (2.5), after some reduction we shall have

$$
\begin{gathered}
\omega_i(t) = (\mathbf{I}^* + \mathbf{N}_i^*)\,\Omega_i(t),\ \Omega_i(t) = \varkappa_i(t)\,[\alpha_i^\circ + c(t)]^{-1}, \\
\mathbf{N}_k^* f(t) = \int_{\tau_r}^{t} f(\tau)\,R_k^\circ(t,\ \tau)\,d\tau \quad (t \in [\tau_r, \tau_{r+1}])
\end{gathered}
\tag{2.6}
$$

where $\mathbf{R}_k{}^0(t, \tau)$ is the resolvent of the kernel

$$K_k{}^\circ(t, \tau) = [c(t) K_1(t, \tau) + \alpha_k{}^\circ K_2(t, \tau)][c(t) + \alpha_k{}^\circ]^{-1}$$

We will now dwell on the algorithm for constructing the characteristic numbers and vector functions of the operator $\mathbf{A}^*$. We will represent the kernel of the operator $\mathbf{A}^*$ and the characteristic vector functions in the form [2]

$$\mathbf{K}(z, \xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_{mn}^{ij} \mathbf{p}_m^i(z) \mathbf{p}_n{}^i(\xi) \tag{2.7}$$

$$\varphi_p(z) = \sum_{k=0}^{\infty} a_{k\,(p)}^l \mathbf{p}_k^l(z) \quad (p = 0, 1, \ldots)$$

$$(\mathbf{p}_k{}^i(z), \quad \mathbf{p}_n{}^j(z)) = \delta_{ij}\delta_{kn}, \qquad \mathbf{p}_k{}^i(z) = P_k{}^*(z)\,\mathbf{i}^i$$

where $\mathbf{p}_k{}^i(z)$ $(k = 0, 1, \ldots)$ is a basis for $L_2([-1, 1], V)$, δ_{mn} is the Kronecker delta and $\mathbf{P}_k{}^*(z)$ is a certain basis for $L_2[-1, 1]$.

On substituting (2.7) into (2.5), we get the following system of algebraic equations for determining the characteristic numbers of the operator $\mathbf{A}^*$ and the coefficients of the expansions of its eigenfunctions in a series in the basis $L_2([-1, 1], V)$:

$$\sum_{n=0}^{\infty} r_{mn}^{ij} a_{n\,(p)}^j = \alpha_p{}^\circ a_{m\,(p)}^i \quad (m = 0, \ldots, i = 1, \ldots n) \tag{2.8}$$

We merely note here that the matrix system (2.8) is symmetric since it follows from the fact that $\mathbf{K}(z, \xi) = \mathbf{K}^T(\xi, z)$ that $r_{mn}{}^{ij} = r_{nm}{}^{ji}$ and the formula holds for all $r_{mn}{}^{ij}$ (the system of orthonormalized Legendre polynomials is selected by the basis $L_2[-1, 1]$):

$$r_{mn}^{ij} = \begin{cases} (-1)^{(m+n-l)/2} R_{mn}^{ij} & (m \text{ and } n \text{ are even: } l = 0; \\ & m \text{ and } n \text{ are odd: } l = 2) \\ (-1)^{(m+n-k)/2} \rho_{mn}^{ij} & (m \text{ is even, } n \text{ is odd: } k = 1; \\ & m \text{ is odd, } n \text{ is even: } k = 1). \end{cases}$$

$$R_{mn}^{ij} = \int_0^\infty f_{mn}^{ij}(u) \cos[(\eta_i - \eta_j) u\lambda^{-1}] du, \quad \rho_{mn}^{ij} = \int_0^\infty f_{mn}^{ij}(u) \times$$
$$\times \sin[(\eta_i - \eta_j) u\lambda^{-1}] du$$
$$f_{mn}^{ij} = [(2m + 1)(2n + 1)]^{1/2} \lambda L(u) u^{-2} J_{m+1/2}(\zeta_i u\lambda^{-1}) \times$$
$$\times J_{n+1/2}(\zeta_j u\lambda^{-1})$$

Let us now consider the actual process of the successive strengthening of a two-layer cylinder with a system of rigid collars. We shall assume that the instants when any two collars are installed on the system are different, that is, the first collar is put on, then the second and so on. In a time interval $t \in [\tau_1, \tau_2]$, we shall have Eq. (2.3), where $i = j = 1$, $\varkappa(z, t) = f^1(z, t)\mathbf{i}^1$. In fact, we obtain a single integral equation with a known right-hand side. After the fitting of the second collar, the operator equation (2.3) will already be equivalent to a system of two equations ($i = j = 1, 2$; $r = 2$) and its right-hand side will contain information on the state of stress and strain of the body obtained in the preceding step

$$\varkappa^1(z,\ t) = f^1(z,\ t) + c(t)\,\mathbf{L}_1{}^{*t}(\tau_1,\ \tau_2)\,q^1(z,\ t) + \\ + \mathbf{L}_2{}^{*t}(\tau_1,\ \tau_2)\,\mathbf{A}_{11}{}^{*}q^{1(z,\ t)}$$

$$\varkappa^2(z,\ t) = f^2(z,\ t) + \mathbf{L}_2{}^{*t}(\tau_1,\ \tau_2)\,\mathbf{A}_{21}{}^{*}q^1(z,\ t)$$

$$\mathbf{L}_l^{*t}(\tau_1, \tau_2)\,w(t) = \int_{\tau_1}^{\tau_2} w(\tau)\,K_l(t, \tau)\,d\tau \qquad (l = 1, 2)$$

where, following after $f^i(z, t)$ $(i = 1, 2)$, the terms determine the distortion of the surface of the cylinder due to the creep of its material. Fitting of a third collar leads to the need to investigate the operator equation, which is equivalent to a system of three two-dimensional integral equations and so on.

In the case when all of the elements of the deformed material are made out of a single material ($\mathbf{L}_1 = \mathbf{L}_2 = \mathbf{L}$, $\nu_1 = \nu_2 = \nu$, $E_1 = E_2 = E$ at the same instant of time ($\tau_1{}^* = \tau_2{}^*$) and all of the collars are fitted simultaneously [see (1.1)], the following assertions can be made:

(1) in a problem concerning the action of a uniform pressure on the internal surface of the cylinder subject to the condition that the tension of all of the collars is equal to zero and that the profiles of the internal surfaces are described by functions which are identically equal to zero, creep has no effect on the stressed state of the body and it is identical to the elastic state;

(2) in contact problems with a rigid insert, the solution can be obtained using the solution of the instantaneous elastic problem on which it is necessary to act with the operator $E(t)(\mathbf{I}+\mathbf{N})E^{-1}(t)$, where $(\mathbf{I}+\mathbf{N}) = (\mathbf{I}-\mathbf{L})^{-1}$.

3. EXAMPLES

Let us consider a two-layer high pressure tube, the layers of which are made at different instants of time from concrete with constant elastic characteristics E and ν and a measure of creep in the form [9]

$$C(t,\ \tau) = (C_0 + Ae^{-\beta\tau})(1 - e^{-\gamma(t-\tau)}) \tag{3.1}$$

Let us put $C_0E = 0.552$, $AE = 4$, $\nu = 0.1$, $\beta = 0.031$ day^{-1}, $\gamma = 0.06$ day^{-1} [5] and, in accordance with the change of variables (2.1), the following values of the parameters are specified:

$$h/(b_i - a_i) = 0{,}15,\ b/(b_i - a_i) = 5,\ a/b = 0{,}8,\ c(t) = 0{,}15$$
$$\theta_0{}^i(t) = 15{,}66,\ P_0{}^i(t) = \sqrt{\zeta_i},\ g^i(z) = 0,\ \delta^i = 0$$
$$\zeta_i = 1,\ \eta_1 = 0,\ \eta_2 = 8,\ \eta_3 = 16\ (i = 1,\ 2,\ 3)$$

that is, collars of the same width with planar profiles of the internal surface are fitted onto the tube without tension at one and the same distance from one another, which is equal to three times the width of the collars.

We shall assume that the internal layer of the cylinder is made at zero instant of time while the outer layer is made 50 days later. The pressure is applied for a further 15 days and the first collar is simultaneously fitted. The second and third reinforcing collars are fitted 13 and 39 days after the first, respectively. For the specified process of the fitting of the cylinder with collars, the dimensionless characteristics of the decisive time parameters take the values $\tau_0 = \tau_1 = 1$, $\tau_2 = 1.2$, $\tau_2 = 1.6$, $\tau_1{}^* = 0.77$ and $\tau_2{}^* = 0$.

In the graphs presented below, we denote the distribution of the contact stresses at the instant of the fitting of a successive collar by the solid lines and the same distributions immediately prior to the fitting of the next collar by broken lines. We also show the changes in the integral characteristics

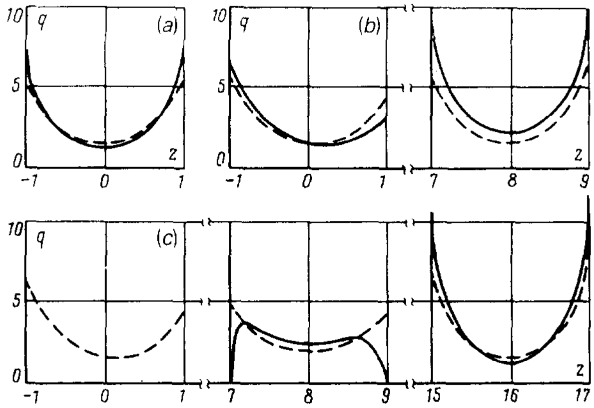

FIG. 2.

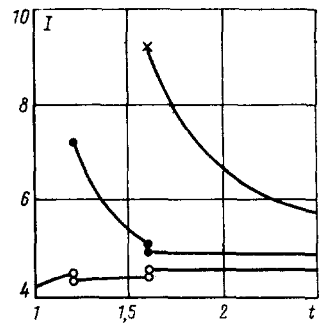

FIG. 3.

$$I_i(t) = \int_{-1}^{1} q^i(z, t)\, dz$$

and label the curves in the graphs for the first, second and third collars by the open circles, solid circles and small crosses, respectively.

The above-mentioned distributions of the contact stresses under one (a), two (b) and three (c) collars are shown in Fig. 2. The dashed line curves in Fig. 2(c) correspond to the limiting distributions when $t \to \infty$. Here, we merely note the principal difference between these distributions of the contact stresses and the distributions which arise when all the collars are fitted simultaneously (the latter are not shown on account of constraints on the volume of graphical material), particularly under the central collar.

The curves for the change in the integral characteristics of the contact stresses with time (Fig. 3) enable one to make a judgement regarding the intensity of the relaxation processes.

We will now consider a further example. Let us assume that the internal layer of the tube is made out of steel with a Young's modulus E_2 and a Poisson's ratio ν_2. This is covered by a layer of viscoelastic ageing polyvinylchloride [10] with elastic characteristics E_1 and ν_1 and a creep measure in the form of (3.1). On the basis of the experimental data for polyvinylchloride [10] and the standard characteristics of steel, we have

$$C_0E_1 = 0.181,\ AE_1 = 0.488,\ \nu_1 = 0.354,\ \nu_2 = 0.3$$
$$\beta = 0.012\ \text{day}^{-1},\ \gamma = 0.315\ \text{day}^{-1},\ E_1E_2^{-1} = 0.016$$

Let us take the following values for the dimensionless quantities:

$$h/(b_i - a_i) = 0.05,\ b/(b_i - a_i) = 5,\ a/b = 0{,}8$$
$$c(t) = 2.81,\ \theta_0{}^i = 24.48,\ P_0{}^i(t) = \sqrt{\zeta_i},\ g^i(z) = 0$$
$$\zeta_i = 1,\ \eta_1 = 0,\ \eta_2 = 4,\ \eta_3 = 8\ (i = 1, 2, 3)$$

that is, unlike the example which was considered earlier, the distance between the collars contracts by a factor of three and their tension is as yet undetermined.

We shall assume that the outer layer is made at the zero instant of time. Fifteen days after the commencement of the measurement of time on the steel cylinder coated with the polymer, the first collar is fitted and the pressure applied. The second and third collars are fitted after 3 and 9 days. Hence, the dimensionless characteristics of the time parameters take their ealier values ($\tau_2{}^*$ is not determined since the internal layer is elastic). In order to identify the curves in the graphs we shall use the notation from the previous example.

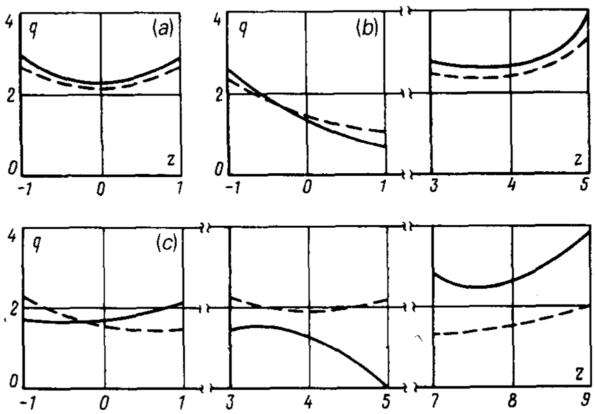

FIG. 4.

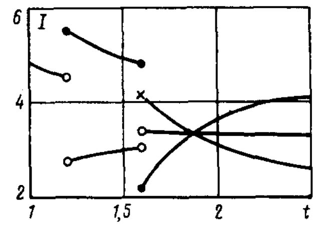

FIG. 5.

The stress distributions under the first collar, which was fitted without tension ($\delta^1 = 0$) and at an instant of time $\tau_1 = \tau_0 = 1$, are shown in Fig. 4(a). It is seen that the process in which the stresses are relaxed for a selected structurally inhomogeneous tube shows up appreciably immediately after the fitting of the first collar. Calculations showed that, in the case when all of the collars were fitted without tension at selected instants of time and with a specified distance between them, separation of the layers occurs under the second collar at the instant when the third collar is fitted. Tension has therefore been given to the second collar.

The stress distributions under the two collars are shown in Fig. 4(b) for $\tau_2 =$ 1, 2 and $\delta^2 = 7.1$.

When the third collar was fitted at the instant $\tau_3 = 1.6$, its tension was chosen such that the integral stress characteristics under the three collars would be close to one another. This tension turned out to negative, that is, $\delta^3 = -3.8$. The commentaries to Fig. 4(c) and Fig. 5 are quite obvious.

In conclusion, it is necessary to point out that the process of the successive fitting of collars on a viscoelastic cylinder leads to qualitatively new phenomena in the behaviour of the characteristics of the contact interaction which do not show up when a system of reinforcing elements is joined to a body at the same time.

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